On a Bound for the Hadamard Product of an M-Matrix and Its Inverse

Li Ching

Department of Mathematics
Leizhou Teacher College
Zhangjiang, Guangdong, P. R. China

and

Chen Ji-Cheng

Department of Computer Science
Zhongshan University
Guangzhou, P.R. China

Submitted by Richard A. Brualdi

ABSTRACT

We give a sharp lower bound for the smallest real eigenvalue $q(A \circ A^{-1})$ of the Hadamard product $A \circ A^{-1}$ of an M-matrix A and its inverse A^{-1} and thus answer positively a conjecture due to Fiedler and Markham.

1. INTRODUCTION

If A is an M-matrix (in this paper, all M-matrices are nonsingular M-matrices), there exists a positive eigenvalue of A equal to $[p(A^{-1})]^{-1}$, where $p(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}. We denote this eigenvalue by $q(A)$.

The Hadamard product of two matrices $A=(a_{ij})$ and $B=(b_{ij})$ of the same dimensions is the matrix $A \circ B=(a_{ij}b_{ij})$.

It has been noted [1] that the Hadamard product $A \circ B^{-1}$ of an M-matrix A and the inverse of an M-matrix B is again an M-matrix. The main purpose

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of this paper is to prove the following inequalities for an $n \times n$ M-matrix A:

$$q(A \circ A^{-1}) > \frac{2}{n} \qquad \text{for} \quad n > 2,$$

$$q(A \circ A^{-1}) = 1 \qquad \text{for} \quad n = 2.$$

We give a positive answer to the conjecture in [2] by combining these two inequalities.

In Section 2, we present some lemmas that are needed for the proofs of the theorems. The main theorem (Theorem 2) is proved in Section 3.

2. LEMMAS

LEMMA 1 [2]. *If $C = (c_{ij})$ is an $n \times n$ M-matrix, diagonally dominant with respect to its columns, i.e.*

$$c_{ii} > \sum_{j \neq i,\, j=1}^{n} |c_{ji}| \qquad \textit{for all } i,$$

then for $C^{-1} = (r_{ij})$ we have

$$r_{ii} > r_{ij} \qquad \textit{for all } i, j,\ i \neq j. \tag{2.1}$$

LEMMA 2. *Suppose A and B are $n \times n$ M-matrices. Then $A \circ B^{-1}$ is an M-matrix.*

Proof. Proposition 3 of [2].

LEMMA 3. *If $0 < x < 1$ and $n > 2$, then*

$$\frac{1 + x}{1 + x + \cdots + x^{n-1}} > \frac{2}{n}. \tag{2.2}$$

Proof. If n is even, we have

$$\frac{1+x+\cdots+x^{n-1}}{1+x}=1+x^2+\cdots+x^{n-2}<\frac{n}{2}.$$

If n is odd, we have

$$\frac{1+x+\cdots+x^{n-1}}{1+x}=1+x^2+\cdots+x^{n-3}+\frac{x^{n-1}}{1+x}<\frac{n}{2}.$$ ■

LEMMA 4. *If $C=(c_{ij})$ is an irreducible $n\times n$ M-matrix and for some real number r, $\sum_{j=1}^{n}c_{ij}>r$ for all i, then $q(C)>r$.*

Proof. There exists a positive eigenvector u of C^T corresponding to $q(C)$; i.e.

$$C^Tu=q(C)u.$$

Let $z=(1,1,\dots,1)^T$. We have $Cz>rz$, and

$$u^TCz=q(C)u^Tz>u^Trz=ru^Tz.$$

Since $u^Tz>0$, the lemma follows. ■

LEMMA 5. *Let $A=(a_{ij})$ be an $n\times n$ M-matrix, and let $A^{-1}=B=(b_{ij})$ be a doubly stochastic matrix. Then*

$$\sum_{j=1}^{n}a_{ij}=1 \qquad \textit{for all } i,$$

and

$$\sum_{i=1}^{n}a_{ij}=1 \qquad \textit{for all } j.$$

Proof. If $e=(1,1,\dots,1)^T$ and we have $A^{-1}e=e$ and $e^TA^{-1}=e^T$, then clearly $Ae=e$ and $e^TA=e^T$, so A is doubly stochastic. ■

The following lemma is quite obvious.

LEMMA 6. *Let p, q be two fixed positive real numbers with $p > q$, and let the real number s satisfy $-q < s \leqslant 0$. Then for any s in the interval $(-q, 0]$, we have*

$$\frac{p+s}{q+s} \geqslant \frac{p}{q} = \min_{s} \frac{p+s}{q+s}. \tag{2.3}$$

3. RESULTS

THEOREM 1. *Suppose that $A = (a_{ij})$ is an irreducible $n \times n$ M-matrix and $A^{-1} = B = (b_{ij})$ is a doubly stochastic matrix with $n > 2$. Then $q(A \circ A^{-1}) > 2/n$.*

Proof. By Lemma 5, we have

$$a_{ii} + \sum_{j=1,\, j \neq i}^{n} a_{ji} = 1 \qquad \text{for all } i. \tag{2.4}$$

Noting that $a_{ij} \leqslant 0$ whenever $i \neq j$, (2.4) implies that

(1) $a_{ii} > 1$ for all i, and
(2) $a_{ii} > \sum_{j=1,\, j \neq i}^{n} |a_{ji}|$ for all i.

Letting $a = \min_i \{a_{ii}\}$, we have $a > 1$. From (2) and Lemma 1, we have $b_{ii} > b_{ij}$ for all i, j, $i \neq j$. The irreducibility of A implies that $A^{-1} = B = (b_{ij})$ is a positive matrix, i.e., $b_{ij} > 0$ for all i, j. Therefore, it is possible to rearrange the ith row of B in the following order:

$$0 \leqslant b_{ij_1} \leqslant b_{ij_2} \leqslant \cdots \leqslant b_{ij_{n-1}} < b_{ii},$$

where the indices j_k, $k = 1, 2, \ldots, n-1$, depend on i. We try to estimate b_{ij_1} as follows. We have

$$1 = \sum_{k=1}^{n} b_{ik} a_{ki} \leqslant a_{ii} b_{ii} + b_{ij_1}(1 - a_{ii})$$

and

$$b_{ij_1} \leqslant \frac{a_{ii}b_{ii}-1}{a_{ii}-1}.$$

Since $i \neq j_1$, we have

$$0 = \sum_{k=1}^{n} b_{ik}a_{kj_1} \leqslant a_{j_1j_1}b_{ij_1} + (1-a_{j_1j_1})b_{ij_2},$$

$$b_{ij_2} \leqslant \frac{a_{j_1j_1}}{a_{j_1j_1}-1}b_{ij_1} \leqslant \frac{a}{a-1}b_{ij_1}. \tag{2.5}$$

To estimate b_{ij_3}, we use the fact that

$$0 = \sum_{k=1}^{n} b_{ik}a_{kj_2} \leqslant a_{j_1j_2}b_{ij_1} + a_{j_2j_2}b_{ij_2} + (1-a_{j_2j_2}-a_{j_1j_2})b_{ij_3}$$

to get the following inequality:

$$b_{ij_3} \leqslant \frac{a_{j_1j_2}b_{ij_1} + a_{j_2j_2}b_{ij_2}}{a_{j_2j_2}-1+a_{j_1j_2}}$$

$$= \frac{a_{j_1j_2} + a_{j_2j_2}b_{ij_2}/b_{ij_1}}{a_{j_1j_2} + (a_{j_2j_2}-1)}b_{ij_1}.$$

Taking $a_{j_2j_2}b_{ij_2}/b_{ij_1}, a_{j_2j_2}-1, a_{j_1j_2}$ to be p, q, s of Lemma 6 respectively, we have, by that lemma and (2.5),

$$b_{ij_3} \leqslant \frac{a_{j_2j_2}}{a_{j_2j_2}-1}b_{ij_2} \leqslant \left(\frac{a}{a-1}\right)^2 b_{ij_1}.$$

By similar discussions, we conclude that

$$b_{ij_k} \leqslant \left(\frac{a}{a-1}\right)^{k-1} b_{ij_1}, \qquad k = 2,3,\ldots,n-1.$$

We go on to estimate b_{ii} as follows:

$$1 = \sum_{j=1}^{n} b_{ij} \leqslant b_{ii} + b_{ij_1}\left[1 + \left(\frac{a}{a-1}\right) + \cdots + \left(\frac{a}{a-1}\right)^{n-2}\right]$$

$$\leqslant b_{ii} + \left[\frac{a_{ii}b_{ii} - 1}{a_{ii} - 1}\right]\left[\frac{1 - \left(\frac{a}{a-1}\right)^{n-1}}{1 - \left(\frac{a}{a-1}\right)}\right]$$

$$\leqslant b_{ii} + (a_{ii}b_{ii} - 1)\left[\left(\frac{a}{a-1}\right)^{n-1} - 1\right].$$

From the above inequality, we get

$$b_{ii}\left\{1 + a_{ii}\left[\left(\frac{a}{a-1}\right)^{n-1} - 1\right]\right\} \geqslant \left(\frac{a}{a-1}\right)^{n-1}.$$

After some calculations, we have

$$b_{ii} \geqslant \frac{a^n}{a_{ii}a^n - a_{ii}a(a-1)^{n-1} + a(a-1)^{n-1}}.$$

Since $a_{ii} \geqslant a > 1$, it can be seen that

$$b_{ii} \geqslant \frac{a^n}{a_{ii}\left[a^n - (a-1)^n\right]}.$$

Therefore, we have

$$\sum_{j=1}^{n} a_{ij}b_{ij} \geqslant a_{ii}b_{ii} + (1 - a_{ii})b_{ij_{n-1}}$$

$$\geqslant a_{ii}b_{ii} + (1 - a_{ii})\frac{a_{ii}b_{ii} - 1}{a_{ii} - 1}\left(\frac{a}{a-1}\right)^{n-2}$$

$$\geqslant \frac{a^n - a^{n-2}(a-1)^2}{a^n - (a-1)^n}$$

$$= \frac{1 - \left(\frac{a-1}{a}\right)^2}{1 - \left(\frac{a-1}{a}\right)^n} > \frac{2}{n} \qquad \text{for all } i.$$

By Lemma 4, we conclude that

$$q(A \circ A^{-1}) > \frac{2}{n}, \qquad n > 2.$$

The proof of Theorem 1 is complete. ■

THEOREM 2. *Let A be an $n \times n$ M-matrix, $n > 2$. Then*

$$q(A \circ A^{-1}) > \frac{2}{n}.$$

Proof. If A is irreducible, then A^{-1} is (strictly) positive and $A \circ A^{-1}$ is again irreducible. By a well-known result of Sinkhorn [3], there exist diagonal matrix D_1 and D_2 with positive diagonal entries such that $D_1A^{-1}D_2$ is doubly stochastic. The matrix $B = D_2^{-1}AD_1$ is again an M-matrix and satisfies $q(A \circ A^{-1}) = q(B \circ B^{-1})$. Since $B \circ B^{-1} = (D_2^{-1}AD_1^{-1}) \circ (D_1A^{-1}D_2) = (D_1D_2^{-1})(A \circ A^{-1})(D_1D_2^{-1})^{-1}$, we get $q(B \circ B^{-1}) > 2/n$ by Theorem 1. This proves the theorem in this case.

Now let A be reducible. We may assume that A has a block upper triangular form (A_{ik}) with irreducible diagonal block A_{ii}, $i = 1, 2, \ldots, s$. Thus A^{-1} is again block upper triangular with diagonal irreducible blocks A_{ii}^{-1}.

Note that

$$q(A \circ A^{-1}) = \min_j q\left(A_{jj} \circ A_{jj}^{-1}\right)$$

and

$$q\left(A_{jj} \circ A_{jj}^{-1}\right) > \frac{2}{n_j},$$

where A_{jj} has order n_j, $j = 1, 2, \dots, s$. By the previous result, we obtain

$$q(A \circ A^{-1}) > \min_j \frac{2}{n_j} = \frac{2}{\max_j n_j} \geqslant \frac{2}{n}.$$

This completes our proof of Theorem 2. ■

For $n = 2$, the best estimate is $q(A \circ A^{-1}) = 2/n = 1$.

If for $n \geqslant 2$, we denote by C the $n \times n$ matrix (c_{ij}) with $c_{11} = c_{22} = \cdots = c_{nn} = a > 1$, $c_{21} = c_{32} = \cdots = c_{n,n-1} = c_{1n} = 1 - a$, where $a = \min_i\{a_{ii}\}$ has been defined in the proof of Theorem 1 with $a > 1$, and the remaining c_{ij} equal to zero, then the matrix C is an M-matrix and

$$q(C \circ C^{-1}) = \frac{1 - \left(\dfrac{a-1}{a}\right)^2}{1 - \left(\dfrac{a-1}{a}\right)^n} > \frac{2}{n}.$$

Theorem 2 demonstrates that the best estimate for $q(A \circ A^{-1})$ is greater than $2/n$. Given $\varepsilon > 0$, choose a sufficiently large n so that $q(C \circ C^{-1}) \leqslant 2/n + \varepsilon$. This shows that the best bound has been obtained.

REFERENCES

1 M. Fiedler, C. R. Johnson, T. Markham, and M. Neumann, A trace inequality for M-matrices and symmetrizability of a real matrix by positive diagonal matrix, *Linear Algebra Appl.* 71:81–94 (1985).

2 M. Fiedler and Thomas L. Markham, An inequality for the Hadamard product of an M-matrix and an inverse M-matrix, *Linear Algebra Appl.* 101:1–8 (1988).

3 R. Sinkhorn, A relationship between arbitrary positive matrices and double stochastic matrices, *Ann. Math. Statist.* 35:876–879 (1964).

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